

# GRASSMANN SECANTS, IDENTIFIABILITY, AND LINEAR SYSTEMS OF TENSORS.

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**ABSTRACT.** For any irreducible non-degenerate variety  $X \subset \mathbb{P}^r$ , we give a criterion for the  $(k, s)$ -identifiability of  $X$ . If  $k \leq s - 1 < r$ , then the  $(k, s)$ -identifiability holds for  $X$  if and only if the  $s$ -identifiability holds for the Segre product  $\text{Seg}(\mathbb{P}^k \times X)$ . Moreover, if the  $s$ -th secant variety of  $X$  is not defective and it does not fill the ambient space, then we can produce a family of pairs  $(k, s)$  for which the  $(k, s)$ -identifiability holds for  $X$ .

## INTRODUCTION

In 1915, in an elegant XIX-century style Italian language ([Ter15, p. 97]), A. Terracini pointed out that the defectiveness of the  $s$ -th secant varieties of a Segre product  $\text{Seg}(\mathbb{P}^k \times V_d)$  between a projective space  $\mathbb{P}^k$  and a Veronese surface  $V_d$ , is related to the fact that the set of all  $\mathbb{P}^k$ 's lying in the span of  $s$  independent points of  $V_d$  does not have the dimension that one can expect, from an obvious count of parameters. That pioneering work (also known as Terracini's second Lemma) was followed by a series of papers by Bronowski, who studied the simultaneous expressions of forms in terms of given powers ([Br33]). Only many decades later, Terracini's analysis has been rephrased in modern terms. In 2001, C. Dionisi and C. Fontanari proved that a Terracini's like result can be formulated by replacing Veronese surface with any irreducible non-degenerate projective variety  $X$  ([DF01, Proposition 1.3]). In analogy with the notion of defectiveness, which is set forth for secant varieties, they utilized the concept of  $(k, s)$ -Grassmann defect that holds for the varieties  $X$  for which the set of all  $\mathbb{P}^k$ 's lying in the span of  $s$  independent points of  $X$  has dimension smaller than the expected one. The Zariski closure  $GS_X(k, s)$  (see Definition 1.3 for more details) of such a set is called  $(k, s)$ -Grassmann secant variety of  $X$  (see also [CC01]). In this setting, Terracini's idea led to the following general result.

**Proposition 0.1.** [DF01, Proposition 1.3] *Let  $X \subset \mathbb{P}^r$  be an irreducible non-degenerate projective variety of dimension  $n$ . Then  $X$  is  $(k, s)$ -defective with defect  $\delta_{k,s}(X) = \delta$  if and only if  $\text{Seg}(\mathbb{P}^k \times X)$  is  $s$ -defective with defect  $\delta_s(\text{Seg}(\mathbb{P}^k \times X)) = \delta$ .*

Actually, it turns out that the relation stated in the previous proposition holds because of the existence of a rational map

$$\Phi : \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) \dashrightarrow GS_X(w, s),$$

where  $w = \min\{s - 1, k\}$ , that can be precisely described, in terms of coordinates.

The map  $\Phi$  determines a general relation between the dimension of  $GS_X(k, s)$  and the dimension of the  $s$ -th secant variety  $\sigma_s(\text{Seg}(\mathbb{P}^k \times X))$  of the Segre embedding of  $\mathbb{P}^k \times X$  into  $\mathbb{P}^{rk+r+k}$  (see Theorem 5.1). Consequently, by studying  $\Phi$ , one

can compute the dimension of  $\sigma_s(\text{Seg}(\mathbb{P}^k \times X))$ , in the case of  $k \geq s - 1$  (see Theorem 5.4). It is worth mentioning that such a result and its consequences endorse Conjecture 5.5 of [AB09b].

Notice that, in the recent preprint [BL11], J. Buczynski and J.M. Landsberg obtain results on the dimension of secant varieties of Segre embeddings ([BL11, Proposition 3.9]). Their method is based on the invariant properties of a rational map  $\pi$ , which is very close to our  $\Phi$ .

Because of the natural way in which the map  $\Phi$  arises, it is easy to guess that it could apply to the study of other properties of Segre products. This is the reason why we decided to assign  $\Phi$  the role of key tool in this paper. We are able to prove, indeed, that the map also provides a link between the *identifiability* of  $X$  and of  $\text{Seg}(\mathbb{P}^k \times X)$  (Theorem 3.1).

A crucial question arising from numerous applications, above all Algebraic Statistics and Signal Processing, is whether or not a set of parameters of a given model is identifiable. From a purely mathematical point of view, this kind of problem can be stated in complete generality. Let  $X \subset \mathbb{P}^r$  be any irreducible non-degenerate projective variety. Given  $s$  distinct points  $P_1, \dots, P_s \in X$ , fix a projective linear subspace  $\Pi \subset \langle P_1, \dots, P_s \rangle$ . How many more  $\mathbb{P}^{s-1}$  containing  $\Pi$  can be found among those that are  $s$ -secants to  $X$ ? When the answer to this question is: “No one besides  $\langle P_1, \dots, P_s \rangle$ ”, then  $\Pi$  is said to be *X-identifiable*. Moreover, if the general  $\mathbb{P}^k$  lying in the span of  $s$  independent points of  $X$  is contained in a unique  $\mathbb{P}^{s-1}$   $s$ -secant to  $X$ , then we say that the *(k, s)-identifiability holds for X* (or equivalently that *X is (k, s)-identifiable*). The identifiability properties are studied, for the case  $k = 0$ , because of their many applications (see, e.g. [LC], [Kru77], [DL06], [AMR09], [CC03], [Com02], [BC11], [CC02], [Mel09], [CO11], [KB09], [ERSS05], [CC11]). In that particular case we will write *s-identifiability* instead of *(0, s)-identifiability*.

Our main contribution to this problem giving rise to new results follows from a direct application of the map  $\Phi$  above, and is summarized in the following two theorems (see Theorems 3.1 and 3.3 respectively).

**Theorem** *Let  $k \leq s - 1 < r$ . The  $(k, s)$ -identifiability holds for  $X$  if and only if the  $s$ -identifiability holds for  $\text{Seg}(\mathbb{P}^k \times X)$ .*

**Theorem** *Let  $s$  be an integer such that  $r > sn + s - 1$  and  $X$  is not  $s$ -defective. Then for all integers  $k > 0$ ,  $k \leq s - 1$  such that*

$$sn + (k + 1)(s - 1 - k) < (k + 1)(r - k)$$

*the  $(k, s)$ -identifiability holds for  $X$ .*

In the particular case in which the variety  $X$  itself is a standard Segre variety  $\text{Seg}(\mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_t))$ , for certain vector spaces  $V_i$ , then the identifiability properties allow one to deduce many peculiar examples on linear systems of tensors  $\mathcal{E} \subset \mathbb{P}(V_1 \otimes \dots \otimes V_t)$  (see Section 4). As an example (see Example 4.6), we can also show that the general linear system of dimension 3 of matrices of type  $4 \times 4$  and rank  $s = 6$  is not identifiable and it is computed by exactly two sets of decomposable tensors.

Actually, Theorem 3.1 can be viewed as the identifiability version of [DF01, Proposition 1.3]. We are persuaded that the rational map  $\Phi$  will be a key tool for

even further investigations on secant varieties and applications. We refer to the remark at the end of Example 5.7, which supports our guess.

After the preliminary Section 1 where we introduce all the needed notions in complete generality for any irreducible non-degenerate projective variety  $Y$ , we devote all of Section 2 to a detailed description of  $\Phi$ . In Section 3 we make use of  $\Phi$  to study the identifiability properties of  $\text{Seg}(\mathbb{P}^k \times X)$  that will be applied in Section 4 for the particular case of linear systems of tensors. Observe that both Theorem 3.1 and Theorem 3.3 quoted above that are the main new theorems of this paper, are obtained by a use of  $\Phi$ . The same belief in the importance of  $\Phi$  urges us to recover [BL11, Prop. 3.9] via a straightforward use of  $\Phi$  (Theorem 5.4). At the end of Section 5, we also show some new, interesting consequences of them, in the particular case of  $X$  being a Segre-Veronese variety.

## 1. PRELIMINARIES, NOTATION AND BASIC DEFINITIONS

Throughout this paper we will always work over an algebraically closed field  $K$  of characteristic 0. All the definitions that we give in this section holds for any irreducible non-degenerate projective variety  $Y$  contained in  $\mathbb{P}^m$ .

Let us recall the classical definition of *secant varieties* and the more modern concept of *Grassmann secant varieties*.

**Definition 1.1.** The  $s$ -th *higher secant variety*  $\sigma_s(Y)$  of  $Y$ , is the Zariski closure of the union of all projective linear spaces spanned by  $s$  distinct points of  $Y$ :

$$\sigma_s(Y) := \overline{\bigcup_{P_1, \dots, P_s \in Y} \langle P_1, \dots, P_s \rangle} \subset \mathbb{P}^m.$$

The expected dimension of  $\sigma_s(Y)$  is

$$(1) \quad \exp \dim \sigma_s(Y) := \min\{s(\dim Y + 1) - 1; m\}.$$

When  $\sigma_s(Y)$  does not have the expected dimension,  $Y$  is said to be *s-defective*, and the positive integer

$$\delta_s(Y) := \exp \dim \sigma_s(Y) - \dim \sigma_s(Y)$$

is called the *s-defect* of  $Y$ .

The fact that  $\sigma_s(Y)$  can have dimension smaller than the expected one, is clearly explained by the well known Terracini's Lemma (the first one). We remark here a consequence that arises when interpreting Terracini's Lemma in terms of fat points (see [CGG11, Section 2]).

**Remark 1.2.** Let  $P_1, \dots, P_s \in Y \subset \mathbb{P}^m$  be generic distinct points, consider the 0-dimensional scheme of  $s$  *2-fat points*  $Z \subset Y$  defined by the ideal sheaf  $\mathcal{I}_Z = \mathcal{I}_{P_1}^2 \cap \dots \cap \mathcal{I}_{P_s}^2 \subset \mathcal{O}_Y$  and denote by  $H(Z, d)$  the Hilbert function of  $Z$  in degree  $d$ .

- i) If  $H(Z, 1) = m + 1$ , then  $\dim \sigma_t(Y) = m$  for all  $t \geq s$ .
- ii) If  $H(Z, 1) = s(\dim Y + 1)$ , then  $\dim \sigma_t(Y) = t(\dim Y + 1) - 1$  for all  $t \leq s$ .

The following definition is due to [CC01].

**Definition 1.3.** Let  $0 \leq k \leq s - 1 \leq m$  be integers and let  $\mathbb{G}(k, m)$  be the Grassmannian of linear  $k$ -spaces contained in  $\mathbb{P}^m$ .

The  $(k, s)$ -Grassmann secant variety of  $Y$ , denoted with  $GS_Y(k, s)$ , is the closure in  $\mathbb{G}(k, m)$  of the set

$$\{\Lambda \in \mathbb{G}(k, m) \mid \Lambda \text{ lies in the linear span of } s \text{ independent points of } Y\}.$$

Notice that, for  $k = 0$ , the Grassmann secant variety  $GS_Y(k, s)$  coincides with the secant variety  $\sigma_s(Y)$ .

The expected dimension of  $GS_Y(k, s)$  is, as always, the minimum between the dimension of the ambient space and the obvious count of parameters obtained by choosing  $s$  points on  $Y$ , and a point in the Grassmannian  $G(k, s - 1)$  (see eg. [CC08]), that is,

$$(2) \quad \exp \dim GS_Y(k, s) = \min\{s(\dim Y) + (k + 1)(s - 1 - k); (k + 1)(m - k)\}.$$

In analogy with the theory of classical secant varieties, we define the  $(k, s)$ -defect of  $Y$  as the integer:

$$\delta_{k,s}(Y) := \exp \dim GS_Y(k, s) - \dim GS_Y(k, s).$$

We end this section by introducing the concept of *identifiability* which will be the core of Sections 3 and 4.

**Definition 1.4.** Fix a linear subspace  $\Pi \subset \mathbb{P}^m$  (possibly a point) and let  $P_1, \dots, P_s \in Y$  be distinct points. We say that  $\Pi$  is *computed by*  $P_1, \dots, P_s \in Y$  if  $\Pi$  belongs to the linear span of the points  $P_i$ 's.

In this case, we say that  $P_1, \dots, P_s$  provide a *decomposition* of  $\Pi$ .

The minimum integer  $s$  for which there exist  $s$  distinct points  $P_1, \dots, P_s \in Y$  such that  $\Pi$  is computed by  $P_1, \dots, P_s$ , is called the *Y-rank* of  $\Pi$ . We indicate it with  $r_Y(\Pi)$ .

**Definition 1.5.** Let  $Y$  and  $\Pi$  be as in Definition 1.4 and let  $s$  be the  $Y$ -rank of  $\Pi$ . We say that  $\Pi$  is *Y-identifiable* if there is a unique set of distinct points  $\{P_1, \dots, P_s\} \subset Y$  whose span contains  $\Pi$ .

**Definition 1.6.** Let  $Y \subset \mathbb{P}^m$  as above. We say that the  $(k, s)$ -*identifiability* holds for  $Y$  if the general element of  $GS_Y(k, s)$  has  $Y$ -rank equal to  $s$  and it is  $Y$ -identifiable.

When  $k = 0$ , we will often omit  $k$  and we will simply say that the *s-identifiability* holds for  $Y$ .

## 2. THE MAP $\Phi$

From now on, with  $X$  we will always denote an irreducible non-degenerate projective variety of dimension  $n$  contained in  $\mathbb{P}^r$ . For any integer  $k \geq 0$ , set  $N = rk + r + k$  and let  $\varphi : \mathbb{P}^k \times X \rightarrow \mathbb{P}^N$  be the Segre embedding of  $\mathbb{P}^k \times X$ . The image of  $\varphi$  is the Segre variety  $Seg(\mathbb{P}^k \times X) \subset \mathbb{P}^N$ .

The aim of this section is to study a projective rational map  $\Phi = \Phi(X, k, s)$  from the  $s$ -th secant variety  $\sigma_s(\mathbb{P}^k \times X)$  of the Segre variety  $Seg(\mathbb{P}^k \times X)$  into the Grassmann secant variety  $GS_X(w, s)$ , where  $w = \min\{s - 1, k\}$ . What we do until Lemma 2.2 can be compared with what is done in [BL11] for the particular case of  $k \leq r$  and  $s \geq k + 1$ . In fact, in [BL11, Corollary 3.6] the authors consider the case of  $k \leq r$  and they introduce a rational map  $\pi$  from the projective space  $\mathbb{P}^N$  to the Grassmannian  $G(k, r)$  whose restriction to  $\sigma_s(\mathbb{P}^k \times X)$  when  $s \geq k + 1$  maps onto  $GS_X(w, s)$ .

We will give a definition of the map  $\Phi$ , in terms of local coordinates. Then, we will show how it allows to link the main secant properties of  $Seg(\mathbb{P}^k \times X)$  with the Grassmann-secant properties of  $X$ .

**Notation 2.1.** For any choice of  $t$  points

$$\mathcal{A}_i = (a_{i,0}, \dots, a_{i,r}) \in K^{r+1}, \quad 1 \leq i \leq t,$$

we denote by  $(\mathcal{A}_1, \dots, \mathcal{A}_t)$  the  $t(r+1)$ -uple

$$(a_{1,0}, \dots, a_{1,r}, \dots, a_{t,0}, \dots, a_{t,r}) \in K^{t(r+1)}.$$

Let  $(\lambda_0, \dots, \lambda_k)$  and  $(x_0, \dots, x_r)$  be sets of homogeneous coordinates for the points  $\Lambda \in \mathbb{P}^k$  and  $P \in X$ , respectively.

Consider the point  $\varphi(\Lambda, P) \in \text{Seg}(\mathbb{P}^k \times X)$ , so that, in coordinates:

$$\varphi(\Lambda, P) = (\lambda_0 x_0, \dots, \lambda_0 x_r, \lambda_1 x_0, \dots, \lambda_1 x_r, \dots, \lambda_k x_0, \dots, \lambda_k x_r).$$

Accordingly with the previous notation, we have:

$$\varphi(\Lambda, P) = (\lambda_0 P, \dots, \lambda_k P).$$

Let  $A$  be a general point in  $\sigma_s(\text{Seg}(\mathbb{P}^k \times X))$ . Then there exist  $s$  distinct points  $\Lambda_1, \dots, \Lambda_s \in \mathbb{P}^k$  and  $s$  distinct points  $P_1, \dots, P_s \in X$  such that  $A \in \langle \varphi(\Lambda_1, P_1), \dots, \varphi(\Lambda_s, P_s) \rangle$ .

Choose a set of homogeneous coordinates  $(a_0, \dots, a_N)$  for  $A$ . By a suitable choice of the homogeneous coordinates  $(\lambda_{i,0}, \dots, \lambda_{i,k})$  of the points  $\Lambda_i$ , we can write:

$$\begin{aligned} A = (a_0, \dots, a_N) &= \varphi(\Lambda_1, P_1) + \dots + \varphi(\Lambda_s, P_s) = \\ &= (\lambda_{1,0} P_1, \dots, \lambda_{1,k} P_1) + \dots + (\lambda_{s,0} P_s, \dots, \lambda_{s,k} P_s). \end{aligned}$$

In the previous notation, this is equivalent to

$$A = (\lambda_{1,0} P_1 + \dots + \lambda_{s,0} P_s, \dots, \lambda_{1,k} P_1 + \dots + \lambda_{s,k} P_s).$$

For any general point  $A$  as above, we set:

$$(3) \quad \Phi(A) = \langle \lambda_{1,0} P_1 + \dots + \lambda_{s,0} P_s, \dots, \lambda_{1,k} P_1 + \dots + \lambda_{s,k} P_s \rangle.$$

Observe that, since  $A$  is general, then the right side of the equality represents a linear space of dimension  $w = \min\{s-1, k\}$ .

We want to show that, in this way, we get indeed a rational map

$$\Phi : \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) \dashrightarrow GS_X(w, s).$$

The map  $\Phi$  is well defined, if

$$(\alpha a_0, \dots, \alpha a_N), \quad \alpha \in K - \{0\}$$

is another set of homogeneous coordinates of  $A$ , then

$$\alpha(a_0, \dots, a_N) = \alpha(\lambda_{1,0} P_1 + \dots + \lambda_{s,0} P_s, \dots, \lambda_{1,k} P_1 + \dots + \lambda_{s,k} P_s),$$

and in this case, obviously,  $\lambda_{1,i} P_1 + \dots + \lambda_{s,i} P_s$  and  $\alpha(\lambda_{1,i} P_1 + \dots + \lambda_{s,i} P_s)$ , for  $0 \leq i \leq k$ , represent the same projective points.

Moreover, if there exist points  $\mathcal{M}_i = (\mu_{i,0}, \dots, \mu_{i,k}) \in \mathbb{P}^k$  and  $Q_i = (y_{i,0}, \dots, y_{i,r}) \in X$  such that

$$A = (a_0, \dots, a_N) = \varphi(\mathcal{M}_1, Q_1) + \dots + \varphi(\mathcal{M}_s, Q_s),$$

we get the following equality of  $(r+1)(k+1)$ -tuples:

$$\begin{aligned} (\lambda_{1,0} P_1 + \dots + \lambda_{s,0} P_s, \dots, \lambda_{1,k} P_1 + \dots + \lambda_{s,k} P_s) &= \\ &= (\mu_{1,0} Q_1 + \dots + \mu_{s,0} Q_s, \dots, \mu_{1,k} Q_1 + \dots + \mu_{s,k} Q_s). \end{aligned}$$

Hence

$$(4) \quad \lambda_{1,i}P_1 + \cdots + \lambda_{s,i}P_s = \mu_{1,i}Q_1 + \cdots + \mu_{s,i}Q_s; \quad i = 0, \dots, k.$$

It follows that  $\Phi$  is consistent.

Next, we give a characterization of points belonging to the inverse image  $\Phi^{-1}(\Pi)$  of a space  $\Pi \in GS_X(k, s)$ .

**Lemma 2.2.** *Let  $w = \min\{k, s-1\}$ ,  $s-1 \leq r$  and take a general point  $\Pi \in GS_X(w, s)$ . Assume  $\Pi \subset \langle P_1, \dots, P_s \rangle$  for  $P_1, \dots, P_s \in X$  distinct points. Let  $B$  be a general element in  $\Phi^{-1}(\Pi)$ . Hence there exist points  $\mathcal{N}_1, \dots, \mathcal{N}_s \in \mathbb{P}^k$  such that*

$$B = \varphi(\mathcal{N}_1, P_1) + \cdots + \varphi(\mathcal{N}_s, P_s).$$

*Proof.* Let us stress, before beginning the proof, the meaning of “general”, in our setting. With “general” point  $\Pi \in GS_X(w, s)$ , (or “general” points of the fiber  $\Phi^{-1}(\Pi)$ ) we mean that, among the points of the Grassmannian (resp. the fiber), the ones for which the statement does not hold consist in a set of zero measure. More specifically, they sit in a proper algebraic subvariety of  $GS_X(w, s)$  (resp.  $\Phi^{-1}(\Pi)$ ). We notice that the generality hypothesis on  $\Pi$  and  $B$  are crucial, for the argument. In fact, for example, if  $\Pi$  is not general in  $GS_X(k, s)$ , we cannot say anything on the number  $s$  of points. Moreover, without the generality hypothesis, we cannot properly define the map  $\Phi$ , itself.

By definition, we know that there are points  $Q_1, \dots, Q_s \in X$  and  $\mathcal{M}_1, \dots, \mathcal{M}_s \in \mathbb{P}^k$ , with  $\mathcal{M}_i = (\mu_{i,0}, \dots, \mu_{i,k})$  for  $i = 1, \dots, s$ , such that

$$\begin{aligned} B &= \varphi(\mathcal{M}_1, Q_1) + \cdots + \varphi(\mathcal{M}_s, Q_s) = \\ &= (\mu_{1,0}Q_1, \dots, \mu_{1,k}Q_1) + \cdots + (\mu_{s,0}Q_s, \dots, \mu_{s,k}Q_s). \end{aligned}$$

Since

$$\begin{aligned} \Phi((\mu_{1,0}Q_1, \dots, \mu_{1,k}Q_1) + \cdots + (\mu_{s,0}Q_s, \dots, \mu_{s,k}Q_s)) &= \\ &= \langle \mu_{1,0}Q_1 + \cdots + \mu_{s,0}Q_s, \dots, \mu_{1,k}Q_1 + \cdots + \mu_{s,k}Q_s \rangle \end{aligned}$$

and

$$\Pi = \langle \lambda_{1,0}P_1 + \cdots + \lambda_{s,0}P_s, \dots, \lambda_{1,k}P_1 + \cdots + \lambda_{s,k}P_s \rangle$$

it follows that each point  $\mu_{1,i}Q_1 + \cdots + \mu_{s,i}Q_s$ , ( $i = 0, \dots, k$ ), lies in the span of the points  $\lambda_{1,j}P_1 + \cdots + \lambda_{s,j}P_s$ , ( $j = 0, \dots, k$ ).

By the definition of  $w$  and by the generality of  $\Pi$ , we may assume that the points  $\lambda_{1,j}P_1 + \cdots + \lambda_{s,j}P_s$ , ( $j = 0, \dots, w$ ) are independent. It follows that, both for  $w = s-1$  and for  $w = k$ , there are coefficients  $\alpha_{i,j} \in K$ , such that

$$\begin{aligned} \mu_{1,i}Q_1 + \cdots + \mu_{s,i}Q_s &= \sum_{j=0}^w \alpha_{i,j}(\lambda_{1,j}P_1 + \cdots + \lambda_{s,j}P_s) = \\ &= \left( \sum_{j=0}^w \alpha_{i,j}\lambda_{1,j} \right) P_1 + \cdots + \left( \sum_{j=0}^w \alpha_{i,j}\lambda_{s,j} \right) P_s \end{aligned}$$

for  $i = 0, \dots, k$ .

So, by setting  $\nu_{h,i} = \left( \sum_{j=0}^w \alpha_{i,j}\lambda_{h,j} \right)$ , we have

$$\mu_{1,i}Q_1 + \cdots + \mu_{s,i}Q_s = \nu_{1,i}P_1 + \cdots + \nu_{s,i}P_s, \quad i = 0, \dots, k.$$

Hence we get:

$$\begin{aligned}
B &= (\mu_{1,0}Q_1 + \cdots + \mu_{s,0}Q_s, \dots, \mu_{1,k}Q_1 + \cdots + \mu_{s,k}Q_s) = \\
&= (\nu_{1,0}P_1 + \cdots + \nu_{s,0}P_s, \dots, \nu_{1,k}P_1 + \cdots + \nu_{s,k}P_s) = \\
&= (\nu_{1,0}P_1, \dots, \nu_{1,k}P_1) + \cdots + (\nu_{s,0}P_s, \dots, \nu_{s,k}P_s) = \\
&= \varphi(\mathcal{N}_1, P_1) + \cdots + \varphi(\mathcal{N}_s, P_s),
\end{aligned}$$

where  $\mathcal{N}_i = (\nu_{i,0}, \dots, \nu_{i,k}) \in \mathbb{P}^k$  for all  $i$ .  $\square$

### 3. SOME CONSEQUENCES ON THE IDENTIFIABILITY OF GENERAL POINTS

The previous construction of the map  $\Phi$  in Section 2, as well as Lemma 2.2, lead to the following analogue of the main theorem in [DF01], for identifiability.

**Theorem 3.1.** *Let  $k \leq s-1 < r$ . The variety  $X$  is  $(k, s)$ -identifiable if and only if  $\text{Seg}(\mathbb{P}^k \times X)$  is  $s$ -identifiable.*

*Proof.* Let  $\Pi$  be a general element of  $GS_X(k, s)$ . If there exist two different sets of distinct points  $\{P_1, \dots, P_s\}, \{Q_1, \dots, Q_s\} \subset X$  such that

$$\Pi \in \langle P_1, \dots, P_s \rangle \quad \text{and} \quad \Pi \in \langle Q_1, \dots, Q_s \rangle,$$

then, by Lemma 2.2, for a general point  $B$  in  $\Phi^{-1}(\Pi)$ , we have points  $\mathcal{M}_i$ 's and  $\mathcal{N}_i$ 's in  $\mathbb{P}^k$ ,  $i = 1, \dots, s$ , with:

$$B = \varphi(\mathcal{M}_1, Q_1) + \cdots + \varphi(\mathcal{M}_s, Q_s) = \varphi(\mathcal{N}_1, P_1) + \cdots + \varphi(\mathcal{N}_s, P_s).$$

Since  $\{P_1, \dots, P_s\} \neq \{Q_1, \dots, Q_s\}$ , we get that  $B$  lies in the span of two distinct sets of points of  $\text{Seg}(\mathbb{P}^k \times X)$ .

Now let  $A$  be a general element of  $\sigma_s(\text{Seg}(\mathbb{P}^k \times X))$ . If

$$A = \varphi(\Lambda_1, P_1) + \cdots + \varphi(\Lambda_s, P_s) = \varphi(\mathcal{M}_1, Q_1) + \cdots + \varphi(\mathcal{M}_s, Q_s)$$

are two different decompositions of  $A$ , then, by the definition of  $\Phi$ ,  $\Pi = \Phi(A)$  lies in the span of the two sets of points  $\{P_1, \dots, P_s\}$  and  $\{Q_1, \dots, Q_s\}$ . It suffices to prove that these two sets of points are distinct.

Since  $A$  is general, we may assume that the two sets of points are both independent. Since:

$$\lambda_{1,i}P_1 + \cdots + \lambda_{s,i}P_s = \mu_{1,i}Q_1 + \cdots + \mu_{s,i}Q_s, \quad i = 0, \dots, k,$$

then  $\{P_1, \dots, P_s\} = \{Q_1, \dots, Q_s\}$  implies  $\lambda_{j,i} = \mu_{j,i}$  for all  $j, i$  (up to re-ordering the  $P_i$ 's and  $Q_i$ 's). This contradicts the fact that the two decompositions of  $A$  are different.  $\square$

**Corollary 3.2.** *If the codimension of  $X$  is bigger than  $s$ , then  $\text{Seg}(\mathbb{P}^{s-1} \times X)$  is  $s$ -identifiable.*

*Proof.* Enough to observe that the general  $s$ -secant  $(s-1)$ -space cuts  $X$  only in  $s$  points, thus it is obvious that a general  $(s-1)$ -space contained in a  $s$ -secant  $(s-1)$ -space, is contained in just one of them!

Then, since under our numerical assumptions we have  $r-n > s$  (hence  $s-1 < r$ ), we may use the previous theorem to get the conclusion.  $\square$

Using Theorem 1.1 of [BC11], which, in turn, is based on the main result of [CGG11] (namely Theorem 4.1), we are able to prove a criterion for the Grassmann identifiability.

**Theorem 3.3.** *Let  $s$  be an integer such that  $r > sn + s - 1$  and  $X$  is not  $s$ -defective. Then for all integers  $k > 0$ ,  $k \leq s - 1$  such that*

$$sn + (k + 1)(s - 1 - k) < (k + 1)(r - k)$$

*the  $(k, s)$ -identifiability holds for  $X$ .*

*Proof.* Theorem 3.1 says that  $(k, s)$ -identifiability holds for  $X$  when  $s$ -identifiability holds for  $\text{Seg}(\mathbb{P}^k \times X)$ . Under our numerical assumptions, the  $s$ -secant variety of  $\text{Seg}(\mathbb{P}^k \times X)$  cannot cover the linear span of  $\text{Seg}(\mathbb{P}^k \times X)$ . Thus we may apply Theorem 1.1 of [BC11], and conclude that  $\text{Seg}(\mathbb{P}^k \times X)$  is  $s$ -identifiable.  $\square$

Indeed, the proof of Theorem 3.1 can be enhanced, to give the following, more precise result:

**Proposition 3.4.** *Let  $w, \Pi, B$  be as in Lemma 2.2. The following two sets:*

$$\mathcal{E}(\Pi) = \{(P_1, \dots, P_s) \in X^s \mid \Pi \subset \langle P_1, \dots, P_s \rangle\}$$

$$\mathcal{E}(B) = \{(P_1, \dots, P_s) \in X^s \mid \exists \mathcal{N}_1, \dots, \mathcal{N}_s \in \mathbb{P}^k \text{ with } B \in \langle (\mathcal{N}_1, P_1), \dots, (\mathcal{N}_s, P_s) \rangle\}$$

*have the same cardinality.*

*Proof.* Almost immediate, following the proof of Theorem 3.1. The unique warning is that the set  $\{P_1, \dots, P_s\}$  that we use in the argument, must be independent. Since  $\Pi, B$  are general, they turn out to be independent for *all* the elements of  $\mathcal{E}(B)$  or  $\mathcal{E}(\Pi)$ , when these sets are finite, and for infinitely many elements, when they are infinite.  $\square$

To be even more precise, the sets  $\mathcal{E}(\Pi)$  and  $\mathcal{E}(B)$  can be endowed with a quasi-projective structure and Lemma 2.2 shows indeed that there exists a birational map  $\mathcal{E}(\Pi) \rightarrow \mathcal{E}(B)$ .

We will not explore this point of view any further, because we do not need it in the sequel.

#### 4. LINEAR SYSTEMS OF TENSORS

In this section, we collect some consequences of the previous theory, trying to explain properly its range of application.

We consider a vector space  $V$  over  $K$  of tensors of type  $n_1 + 1, \dots, n_t + 1$ , namely  $V = V_1 \otimes \dots \otimes V_t$  where  $V_i$  is a vector space of dimension  $n_i + 1$ , for  $i = 1, \dots, t$ . A *linear system* of tensors is just a linear subspace of  $V$ . In the projective setting, tensors of type  $n_1 + 1, \dots, n_t + 1$ , up to scalar multiplication, determine a projective space  $\mathbb{P}^M$ , where  $M = (\prod_{i=1}^t (n_i + 1)) - 1$ . A linear system of tensors is a linear subspace  $\mathcal{E}$  of  $\mathbb{P}^M$ .

We take the *dimension* of  $\mathcal{E}$  to be the *projective* dimension of the linear subspace associated to  $\mathcal{E}$  (i.e. the affine dimension, minus 1).

Inside the space of tensors, there is the subvariety  $X$  of *decomposable tensors*, which corresponds to the Segre embedding  $X = \text{Seg}(\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_t}) \subset \mathbb{P}^M$ .

**Definition 4.1.** We say that the linear system  $\mathcal{E}$  of tensors is *computed* by  $s$  decomposable tensors  $P_1, \dots, P_s \in X$  if for all  $P \in \mathcal{E}$  there are scalars  $a_1, \dots, a_s$  such that:

$$P = a_1 P_1 + \dots + a_s P_s.$$



Geometrically, this means that the linear space associated to  $\mathcal{E}$  lies in the span of the points  $P_1, \dots, P_s$ .

We say that  $\mathcal{E}$  has *rank*  $s$  if  $s$  is the minimum such that there are  $s$  tensors in  $X$  which compute  $\mathcal{E}$ .

We say that a linear system  $\mathcal{E}$  of rank  $s$  is *identifiable* if there exists a unique set of  $s$  decomposable tensors, that compute  $\mathcal{E}$ .

We say that tensors of type  $n_1 + 1, \dots, n_t + 1$  are  $(k, s)$ -*identifiable* if the general linear system  $\mathcal{E}$  of such tensors, of dimension  $k$  and rank  $s$ , is identifiable.

It is immediate to see that the previous terminology is consistent with the general terminology of the paper, once one considers the linear subspace associated to a linear system (see also Definition 1.1 of [BL11]).

The map  $\Phi$  constructed in the previous sections maps a tensor  $P$  of type  $k + 1, n_1 + 1, \dots, n_t + 1$  to a linear system of dimension  $k$  of tensors of type  $n_1 + 1, \dots, n_t + 1$ . Roughly speaking, the map takes the tensor  $T$  to the linear space generated by its  $k + 1$  slices along the first direction.

Thus, all the results in the previous section apply to the identifiability of linear systems of tensors. In particular, for instance, we see that:

**Remark 4.2.**

- (i) The general linear systems of dimension  $k$  of tensors of type  $n_1 + 1, \dots, n_t + 1$  has rank  $s$  if and only if  $s$  is the minimum such that the secant variety  $\sigma_s(\mathbb{P}^k \times \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_t})$  covers the projective space  $\mathbb{P}^N$ ,  $N = (M + 1)(k + 1) - 1$ .
- (ii) There are exactly  $q$  sets of decomposable tensors that compute a general linear system of tensors of type  $n_1 + 1, \dots, n_t + 1$  if and only if there are exactly  $q$  decomposable tensors that compute a general tensors of type  $k + 1, n_1 + 1, \dots, n_t + 1$ .
- (iii) Tensors of type of type  $n_1 + 1, \dots, n_t + 1$  are  $(k, s)$ -identifiable if and only if tensors of type  $k + 1, n_1 + 1, \dots, n_t + 1$  are  $s$ -identifiable.

Let us see how the previous remarks allows to translate some known facts about tensors to facts about linear systems of tensors. The next two examples are actually consequences of the gluing of the main results of [CGG11] and [BC11].

**Example 4.3.** *For  $m > 4$ , the general linear pencil of tensors of type  $2 \times \dots \times 2$ , ( $m$ -times) has rank  $\lceil 2^m / (m + 1) \rceil$ .*

*The general linear pencil as above, of rank  $s \leq 2^{m-1} / m$ , is identifiable.*

The first fact follows from the main result in [CGG11] (namely Theorem 4.1) that computes the dimension of the secant varieties of the Segre embedding of  $\mathbb{P}^1 \times \dots \times \mathbb{P}^1$ . The value  $\lceil 2^m / (m + 1) \rceil$  corresponds to the order of the secant variety that fills the ambient space. This result together with Remark 4.2 (i) proves the first fact.

The second fact follows from Remark 4.2 (ii) and the main result of [BC11] (Theorem 1.1) which shows that the Segre product of  $(m + 1) > 5$  copies of  $\mathbb{P}^1$ 's is  $k$ -identifiable as soon as  $\lceil 2^m / (m + 1) \rceil$ .

**Example 4.4.** *The general linear pencil of tensors of type  $2 \times 2 \times 2 \times 2$  has rank 6.*

*The general linear pencil of tensors of type  $2 \times 2 \times 2 \times 2$ , of rank  $s < 5$ , is identifiable.*

*The general linear pencil of tensors of type  $2 \times 2 \times 2 \times 2$ , of rank 5, is NOT identifiable: it is computed by exactly two sets of decomposable tensors.*

Just use the main results in [CGG11] (Theorem 4.1) that gives the order of the secant variety of the Segre variety that covers the ambient space, and Proposition 4.1 of [BC11] that explicitly says that the product of 5 copies of  $\mathbb{P}^1$  is not 4-identifiable and moreover that through a general point of the fifth secant variety one finds exactly two 5-secant, 4-spaces. These two results together with our Remark 4.2 leads to the example.

There are also results for linear systems of matrices, which, as far as we know, cannot be found in the classical literature hence we quote the next two examples as surprising new facts on matrices.

**Example 4.5.** *The general linear system of rank  $s$  and dimension  $c-1$ , of matrices of type  $a \times b$ , with  $a \leq b \leq c$ , is identifiable, as soon as  $s \leq ab/16$ .*

It follows from the main result in [CO11] (Theorem 1.1) applied to our Remark 4.2 (iii). In fact [CO11, Theorem 1.1] states that the general tensor of  $V_1 \otimes V_2 \otimes V_3$  of rank  $k$  has a unique decomposition if  $k \leq 2^{\alpha+\beta-2}$  where  $\alpha, \beta$  are the maximal integers such that  $2^\alpha \leq \dim V_1$  and  $2^\beta \leq \dim V_2$  and  $\dim V_1 \leq \dim V_2 \leq \dim V_3$ .

**Example 4.6.** *The general linear system of dimension 3 of matrices of type  $4 \times 4$  has rank 7.*

*The general linear system of dimension 3 of matrices of type  $4 \times 4$  and rank  $s < 6$  is identifiable.*

*The general linear system of dimension 3 of matrices of type  $4 \times 4$  and rank  $s = 6$  is NOT identifiable: it is computed by exactly two sets of decomposable tensors.*

Use the main results in [AOP09], and [CO11, Theorem 1.3]. In particular [AOP09, Example 3.18] shows that the Segre embedding of  $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$  is never defective, then its 7-th secant variety fills the ambient space. While [CO11, Theorem 1.3] says that a general tensor in  $\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4$  of rank 6 has exactly two decompositions. These two results glued together with our Remark 4.2 give the example.

Tons of similar results, about the identifiability of linear systems of tensors, can be found by rephrasing, from the point of view of Remark 4.2 the examples that the reader can find in [Kru77], [DL06], [Lic85], [CGG11], [BC11], [AOP09], [CO11], [BCO] etc.

We will not expound further on this subject.

## 5. SOME CONSEQUENCES ON THE DIMENSION OF SECANT VARIETIES OF SEGRE VARIETIES

The construction introduced with the map  $\Phi$  in Section 2, as well as the obvious remark at the beginning of the proof of Corollary 3.2, is indeed useful for the study of many aspects of Segre products. In this section, we would like to point out how the study of the map can be used to determine the dimension of some secant variety.

Notice that the results of our Theorem 5.4 correspond to Corollary 3.2 and Proposition 3.9 of [BL11]. The method of J. Buczyński and J.M. Landsberg is based on the invariant properties of a rational map  $\pi$  (see [BL11] [Corollary 3.6]), which is very close to our  $\Phi$ .

We add this section because we would like to re-organize the results, showing how they follow from an elementary coordinate-based examination of the map  $\Phi$ .

**Theorem 5.1.** *Assume, as always,  $w = \min\{k, s-1\}$  and  $s-1 \leq r$ . Then we have:*

$$\dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = \dim GS_X(w, s) + (w+1)(k+1) - 1.$$

*Proof.* Let  $\Pi$  be a general element of  $GS_X(w, s)$ , that is,  $\Pi$  is a  $w$ -space contained in  $\langle P_1, \dots, P_s \rangle$ , where the  $P_i$  are independent points of  $X$ .

If we prove that  $\dim \Phi^{-1}(\Pi) = (w+1)(k+1) - 1$ , we are done.

Even if  $w < k$ , we can fix scalars  $\lambda_{i,j} \in K$ , with  $i = 1, \dots, s$  and  $j = 0, \dots, k$ , such that

$$\Pi = \langle \lambda_{1,0}P_1 + \dots + \lambda_{s,0}P_s, \dots, \lambda_{1,k}P_1 + \dots + \lambda_{s,k}P_s \rangle.$$

Consider the points  $\Lambda_i = (\lambda_{i,0}, \dots, \lambda_{i,k}) \in \mathbb{P}^k$ , and let

$$(5) \quad A = \varphi(\Lambda_1, P_1) + \dots + \varphi(\Lambda_s, P_s) \in \sigma_s(\text{Seg}(\mathbb{P}^k \times X)).$$

Obviously  $A \in \Phi^{-1}(\Pi)$  and, for a general choice of the scalars,  $A$  will be a general point of  $\Phi^{-1}(\Pi)$ .

Since  $s \leq r+1$ , without loss of generality, we may assume that the  $P_i$  are coordinate points, say

$$P_1 = (1, 0, \dots, 0), P_2 = (0, 1, \dots, 0), \dots, P_s = (0, \dots, 0, 1, \dots, 0).$$

With this choice of coordinates, it is easy to see that

$$\Phi(A) = \langle (\lambda_{1,0}, \lambda_{2,0}, \dots, \lambda_{s,0}, 0, \dots, 0), \dots, (\lambda_{1,k}, \lambda_{2,k}, \dots, \lambda_{s,k}, 0, \dots, 0) \rangle,$$

Now, fix another general point  $B \in \Phi^{-1}(\Pi)$ . By Lemma 2.2, we know that there are points  $\mathcal{M}_i = (\mu_{i,0}, \dots, \mu_{i,k}) \in \mathbb{P}^k$  with

$$B = \varphi(\mathcal{M}_1, P_1) + \dots + \varphi(\mathcal{M}_s, P_s)$$

and so:

$$\Phi(B) = \langle (\mu_{1,0}, \mu_{2,0}, \dots, \mu_{s,0}, 0, \dots, 0), \dots, (\mu_{1,k}, \mu_{2,k}, \dots, \mu_{s,k}, 0, \dots, 0) \rangle.$$

Since  $\Phi(A) = \Phi(B)$ , in the case  $w = k \leq s-1$ , it follows that each point  $(\mu_{1,i}, \mu_{2,i}, \dots, \mu_{s,i}, 0, \dots, 0)$ , ( $i = 0, \dots, k$ ), lies in the span of the  $k+1$  points  $(\lambda_{1,j}, \lambda_{2,j}, \dots, \lambda_{s,j}, 0, \dots, 0)$ , ( $j = 0, \dots, k$ ).

In case  $w = s-1 < k$ , each point  $(\mu_{1,i}, \mu_{2,i}, \dots, \mu_{s,i}, 0, \dots, 0)$ , ( $i = 0, \dots, k$ ), lies in the span of  $w+1$  independent points among the  $k+1$  points  $(\lambda_{1,j}, \lambda_{2,j}, \dots, \lambda_{s,j}, 0, \dots, 0)$ , ( $j = 0, \dots, k$ ), and we may assume that these  $w+1$  independent points are  $(\lambda_{1,j}, \lambda_{2,j}, \dots, \lambda_{s,j}, 0, \dots, 0)$ , with  $j = 0, \dots, w$ .

In other words, there exist  $(w+1)(k+1)$  elements  $\alpha_{i,j} \in K$  s.t.

$$(\mu_{1,i}, \mu_{2,i}, \dots, \mu_{s,i}, 0, \dots, 0) = \sum_{j=0}^w \alpha_{i,j} (\lambda_{1,j}, \lambda_{2,j}, \dots, \lambda_{s,j}, 0, \dots, 0),$$

where  $i = 0, \dots, k$ .

Equivalently, the following linear system

$$\begin{pmatrix} M & 0 & 0 & \dots & 0 & 0 \\ 0 & M & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0 & M \end{pmatrix} \begin{pmatrix} \alpha_{0,0} \\ \dots \\ \alpha_{0,k} \\ \alpha_{1,0} \\ \dots \\ \alpha_{1,k} \\ \dots \\ \alpha_{k,0} \\ \dots \\ \alpha_{k,k} \end{pmatrix} = \begin{pmatrix} \mu_{1,0} \\ \dots \\ \mu_{s,0} \\ \mu_{1,1} \\ \dots \\ \mu_{s,1} \\ \dots \\ \mu_{1,k} \\ \dots \\ \mu_{s,k} \end{pmatrix}$$

where  $M = \begin{pmatrix} \lambda_{1,0} & \dots & \lambda_{1,k} \\ \lambda_{2,0} & \dots & \lambda_{2,k} \\ \dots & \dots & \dots \\ \lambda_{s,0} & \dots & \lambda_{s,k} \end{pmatrix}$ , has solutions. Since  $A$  is general, the rank of the coefficient matrix of this linear system is  $(w+1)(k+1)$ .

Now, since

$$\begin{aligned} B &= \varphi(\mathcal{M}_1, P_1) + \dots + \varphi(\mathcal{M}_s, P_s) \\ &= (\mu_{1,0}P_1, \dots, \mu_{1,k}P_1) + \dots + (\mu_{s,0}P_s, \dots, \mu_{s,k}P_s) \\ &= (\mu_{1,0}, \mu_{2,0}, \dots, \mu_{s,0}, 0, \dots, 0, \mu_{1,1}, \mu_{2,1}, \dots, \mu_{s,1}, 0, \dots, 0, \\ &\quad \dots, \mu_{1,k}, \mu_{2,k}, \dots, \mu_{s,k}, 0, \dots, 0), \end{aligned}$$

it immediately follows that the dimension of  $\Phi^{-1}(\Pi)$  is  $(w+1)(k+1) - 1$ .  $\square$

The previous argument shows that the map  $\Phi$  has positive dimensional fibers, in general. It is interesting to observe that, nevertheless, the structure of the fibers yields that the identifiability of  $\Phi(A)$  implies the identifiability of  $A$ .

As an easy consequence of Theorem 5.1, we get the following Terracini-type theorem (proved in [DF01]):

**Corollary 5.2.** *Let  $k \leq s-1 < r$ . Then  $X$  is  $(k, s)$ -defective with defect  $\delta_{k,s}(X) = \delta$  if and only if  $\text{Seg}(\mathbb{P}^k \times X)$  is  $s$ -defective with defect  $\delta_s(\text{Seg}(\mathbb{P}^k \times X)) = \delta$ .*

*Proof.* By Theorem 5.1, and a direct computation we have

$$\begin{aligned} \dim(\sigma_s(\text{Seg}(\mathbb{P}^k \times X))) - \dim(GS_X(k, s)) &= k^2 + 2k \\ &= \exp \dim(\sigma_s(\text{Seg}(\mathbb{P}^k \times X))) - \exp \dim(GS_X(k, s)) \end{aligned}$$

and we are done.  $\square$

Next, we get some results about the defectivity or non-defectivity of the  $s$ -th higher secant variety of  $\text{Seg}(\mathbb{P}^k \times X)$ .

**Lemma 5.3.** *For  $s-1 < k$ , and  $s-1 \leq r$ , we have*

$$\dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = \min\{s(k+n+1)-1; s(k+r-s+2)-1\}$$

*Proof.* By Theorem 5.1 we get

$$\dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = \dim GS_X(s-1, s) + s(k+1) - 1.$$

Since it is well known, (see, for instance, [CC08, Section 2]), that the dimension of  $GS_X(s-1, s)$  is the smallest between  $sn$  and the dimension of the Grassmannian  $\mathbb{G}(s-1, r)$ , the conclusion easily follows.  $\square$

**Theorem 5.4.** *Let  $X \subset \mathbb{P}^r$  be an irreducible non-degenerate projective variety of dimension  $n$ .*

(i) *If  $s - 1 \geq r$ , then*

$$\sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = \mathbb{P}^N,$$

*so  $\sigma_s(\text{Seg}(\mathbb{P}^k \times X))$  is not defective.*

(ii) *Let  $s - 1 < \min\{r; k\}$ ;*

(a) *if  $s - 1 \leq r - n$ , then*

$$\dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = s(k + n + 1) - 1,$$

*and  $\sigma_s(\text{Seg}(\mathbb{P}^k \times X))$  is not defective;*

(b) *if  $s - 1 > r - n$ , then*

$$\dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = s(k + r - s + 2) - 1,$$

*and  $\sigma_s(\text{Seg}(\mathbb{P}^k \times X))$  is defective.*

(iii) *If  $s - 1 = k < r$ , then*

$$\dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = \min\{s(k + n + 1) - 1, N\},$$

*and  $\sigma_s(\text{Seg}(\mathbb{P}^k \times X))$  is not defective.*

(iv) *If  $k < s - 1 < r$ , then*

$$\dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = \dim GS_X(k, s) + k^2 + 2k.$$

*Proof.* (i) It is enough to prove this case for  $s - 1 = r$ . Let  $P_1, \dots, P_s$  be independent points in  $X$ . We may assume that  $P_1 = (1, 0, \dots, 0), P_2 = (0, 1, \dots, 0), \dots, P_s = (0, \dots, 0, 1)$ . Hence for a general point  $A = (\lambda_{1,0}, \lambda_{2,0}, \dots, \lambda_{s,0}, \dots, \lambda_{1,k}, \lambda_{2,k}, \dots, \lambda_{s,k}) \in \mathbb{P}^N$  we have  $A = \varphi(\Lambda_1, P_1) + \dots + \varphi(\Lambda_s, P_s)$ , where  $\Lambda_i = (\lambda_{i,0}, \dots, \lambda_{i,k})$ .

(ii) By Lemma 5.3 we immediately get the dimensions of  $\sigma_s(\text{Seg}(\mathbb{P}^k \times X))$  both in case (a) and in case (b).

Since in case (ii)(a) we get  $N > s(k + n + 1) - 1$ , it follows that  $\exp \dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = s(k + n + 1) - 1$ , and so in this case  $\sigma_s(\text{Seg}(\mathbb{P}^k \times X))$  is not defective.

In case (ii)(b) we have

$$s(k + n + 1) - 1 - \dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = s(n - r + s - 1) > 0,$$

$$N - \dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = (r - s + 1)(k - s + 1) > 0.$$

Hence  $\dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) < \exp \dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X))$ .

(iii) For  $s - 1 = k$ , we have  $s(k + r - s + 2) - 1 = N$ , hence by Lemma 5.3 we get the conclusion.

(iv) Obvious from Theorem 5.1.

□

If  $k = r - n$ , by applying the theorem above we get the following interesting result.

**Corollary 5.5.** *If  $k = r - n$ , then  $\sigma_s(\text{Seg}(\mathbb{P}^k \times X))$  is never defective.*

*Proof.* First assume  $s - 1 = k$ . By Theorem 5.4 (iii) we get

$$\dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = \min\{s(n + s) - 1, s(r + 1) - 1\}.$$

Since  $r - n = s - 1$  we have

$$s(n + s) - 1 = s(r + 1) - 1,$$

$$s(n + s) - 1 = (k + 1)(r + 1) - 1 = N,$$

$$s(r + 1) - 1 = s(k + n + 1) - 1,$$

and so

$$\dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = N = s(k + n + 1) - 1 = s(\dim \text{Seg}(\mathbb{P}^k \times X) + 1) - 1.$$

Now assume that  $s \neq k + 1$ . In this case, by Remark 1.2, we get:

- for  $s > k + 1$ ,  $\dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = N$  ;
- for  $s < k + 1$ ,  $\dim \sigma_s(\text{Seg}(\mathbb{P}^k \times X)) = s(k + n + 1) - 1$ ,

and the conclusion follows.  $\square$

Next two examples, which are the only new spots in this section, show how the previous analysis of the map  $\Phi$ , and the properties listed above, allow us to settle some interesting facts about Segre varieties and tensors.

**Example 5.6.** Let  $Y$  be the Segre-Veronese embedding of  $\mathbb{P}^{\binom{n+1}{2}} \times \mathbb{P}^n$  via divisors of bi-degree  $(1, 2)$ . Then  $\sigma_s Y$  is never defective. In fact, let  $X \subset \mathbb{P}^{\binom{n+2}{2}-1}$  be the 2-uple Veronese embedding of  $\mathbb{P}^n$ . Since

$$Y = \text{Seg}(\mathbb{P}^{\binom{n+1}{2}} \times X).$$

and since  $\binom{n+1}{2} = \binom{n+2}{2} - 1 - n$ , then from Corollary 5.5 we get the conclusion.

We like to stress here that this proves one case of the Conjecture 5.2 of [AB09a] (see also the more general Conjecture 5.5 of [AB09b]).

**Example 5.7.** Let  $Y$  be the Segre-Veronese embedding of  $\mathbb{P}^k \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$  via divisors of multi-degree  $(1, d_1, \dots, d_t)$ . If  $k = \prod_{i=1}^t \binom{n_i + d_i}{d_i} - \sum_{i=1}^t n_i - 1$ , Corollary 5.5 implies that  $\sigma_s(Y)$  is never defective.

We end the paper with the following interesting remark pointed out by the anonymous referee that we thank for having shared it with us. In the previous example, if we consider the particular case of  $Y$  being the standard Segre embedding obtained by taking all  $d_i$ 's equal to 1, then we get kind of tensors of boundary format in the sense of Hyperdeterminants of Gelfand, Kapranov and Zelevinsky [GKZ94][pp. 444–445].

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